Peierls-Boltzmann equation for ballistic deposition

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We consider nonlinear stochastic field equations. Going over to a Fokker-Planck description, we construct a self-consistent expansion around a model evolution equation. In second order the equation for the two-point function resembles the Peierls-Boltzmann equation for the average number of phonons, but involves also the unknown characteristic frequency function. Within the same expansion we obtain an equation for that function too. The two coupled equations are studied specifically for the case of ballistic deposition. We show how to obtain the exact asymptotic solution of the two coupled nonlinear integral equations obtained in second order. Higher orders are also discussed. [S1063-651X(98)03005-0]

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I. INTRODUCTION

Many interesting phenomena in condensed-matter physics are described by nonlinear equations driven by random forces. A long list of examples includes turbulence, critical dynamics, and the dynamics of interacting polymers. Many of these examples are difficult, not only because of their nonlinearity, but for other problems such as convergence difficulties in turbulence and topological difficulties in polymers. However, the problem of nonlinear deposition theory, as described by the Kardar-Parisi-Zhang (KPZ) [1] equation, is a benign example, showing fully nonlinear behavior, but without other difficulties. It has a quadratic nonlinearity, which leads to manageable mathematics. This feature belongs to physical systems that are dissipative; Hamiltonian systems are notably more difficult. In a previous publication we gave a brief description of a self-consistent expansion that allowed a direct calculation of the indices that characterize the correlation functions of that equation [2]. The method is based on going over from the KPZ equation in Langevin form to a Fokker-Planck form and constructing a self-consistent expansion of the distribution for the field concerned and hence the required observables. The method produced useful equations at second order of nonlinear coupled integral equations, which can be solved exactly in the asymptotic limit to yield exponents governing the steady-state behavior and the time evolution, although this latter problem is not dealt with in detail in this paper. The present article gives a detailed explanation of the method not only to show how and why it works for the KPZ equation, but also to show how other systems can be studied.

The article is organized as follows. In Sec. II we obtain the Fokker-Planck equation derived from a generic Langevin-like field equation. The idea of choosing a model system that is soluble and approximates the nonsoluble system, is described in Sec. III. A second-order expansion around such a model is used in Sec. IV to obtain a Peierls-Boltzmann equation for the static two-point function in terms of an undetermined characteristic frequency function. Dynamical arguments are used in Sec. V to derive an equation for the characteristic frequency. The resultant two coupled nonlinear integral equations for the static two-point function and the characteristic frequency are discussed in Sec. VI. We show how to obtain the exact exponents describing the asymptotic small-q behavior of the two functions. Higher-order expansions are considered in Sec. VII. It is shown that a strong-coupling consistent expansion is possible only if a certain scaling relation between the exponents characterizing the two-point function and the characteristic frequency is obeyed. It is very interesting that this scaling relation is identical to the one that follows from the dynamical arguments. The construction of the small-q asymptotic solution is detailed in the Appendix.

II. NONLINEAR EQUATIONS

We study a field $h(\mathbf{r},t)$, $h_k(t)$, or $h_{k\omega}$ in progressive Fourier transforms, which satisfies the equation

$$\frac{\partial h_k}{\partial t}-\nu_k h_k+\sum_{j,l} M_{kjl}h_j h_l+\sum_{j,l,m} N_{kjlm}h_j h_l h_m+\cdots=\eta_k, \tag{2.1}$$

where usually M,N have $k+j+l=0$, $k+j+l+m=0$, etc. The variable η_k is noise driving the equation and usually has the form

$$\langle \eta_k\rangle=0, \tag{2.2}$$

$$\langle \eta(\mathbf{r},t)\,\eta(\mathbf{r}',t')\rangle=2D(\mathbf{r}-\mathbf{r}')\,\delta(t-t'), \tag{2.3}$$

or in the simplest case

$$2D_0\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'). \tag{2.4}$$

An example is the KPZ equation for a height h,

$$\frac{\partial h}{\partial t}-\nu\nabla^2 h+g(\boldsymbol{\nabla} h)^2=\eta(\mathbf{r},t). \tag{2.5}$$

Equation (2.2) means that the height h is measured relative to the mean. The linear equation

$$\frac{\partial h}{\partial t} - \nu \nabla^2 h = \eta \tag{2.6}$$

already contains much interesting physics [3], but when the nonlinear term is included the correlation function

$$\langle h(\mathbf{r},t) h(\mathbf{r}',t') \rangle = \phi(\mathbf{r}-\mathbf{r}',t-t') \tag{2.7}$$

has a composite behavior, which leads to the breakdown of q space into regions characterized by the different behavior of its Fourier transform. At low q it exhibits power-law behavior. In Fourier variables the KPZ equation is

$$\frac{\partial h_q}{\partial t} + \nu q^2 h_q + \frac{g}{\sqrt{\Omega}} \sum_l l\cdot(q-l) h_l h_{q-l} = \eta_q(t), \tag{2.8}$$

i.e.,

$$\nu_q = \nu q^2, \tag{2.9}$$

$$M_{qlm} = \delta_{q+l+m}(l\cdot m) \frac{g}{\sqrt{\Omega}}. \tag{2.10}$$

Here Ω is the volume of the system and passing to large Ω, the Fourier components run into the continuum. From Eq. (2.1) we can pass to Liouville's equation

$$\frac{\partial P}{\partial t} + \sum_q \frac{\partial}{\partial h_{-q}} \left[\nu_q h_q + \sum_{l,m} M_{qlm} h_l h_m + \eta_{-q} \right] P = 0, \tag{2.11}$$

where P is the probability that $h_q(t)$ has the value h_q:

$$P = \prod_q \delta(h_q(t) - h_q). \tag{2.12}$$

When η satisfies Eq. (2.4), i.e., fluctuates locally in time, the average of P over η satisfies

$$\frac{\partial}{\partial t}\langle P\rangle + \sum_q \frac{\partial}{\partial h_{-q}} \left[D_{0q} \frac{\partial}{\partial h_q} + \nu_q h_{-q} + \sum_{l,m} M_{qlm} h_l h_m \right] \langle P \rangle$$
$$\equiv \frac{\partial}{\partial t}\langle P\rangle + L\langle P\rangle = 0. \tag{2.13}$$

This can easily be shown by expanding Eq. (2.11) in η and then averaging term by term, resuming, or by many other methods: It is a standard result.

Henceforth $\langle P\rangle$ will be referred to as P. An exact proof has been given that a steady state exists for the KPZ equation (in the case $D_{0q} = D_0$) in one dimension [4]. (The steady state is a simple Gaussian.) In higher dimensions the existence of a steady state follows from numerical simulations. Therefore, we assume in this paper without proof that there exists a steady-state solution

$$\frac{\partial P}{\partial t} = 0 \tag{2.14}$$

in any number of dimensions.

In general, it is difficult to envisage circumstances when this will not be the case. Even when, in the case of fluid turbulence, there are well-established long-lived fluctuations, over a longer period again one expects Eq. (2.14) to hold. Thus our problem is now reduced to

$$LP = 0. \tag{2.15}$$

In the next section we will propose a systematic approach to a solution and in Sec. IV give details of the solution.

III. MODELS AND EXPANSIONS

In the following we describe the motivation for our self-consistent expansion. Suppose the nonlinear term in Eq. (2.8) was modeled as if it were an addition to η, i.e., suppose

$$\left\langle \sum_{l,m} M_{qlm} h_l(t) h_m(t) \sum_{l',m'} M_{-q'-l'-m'} h_{-l'}(0) h_{-m'}(0) \right\rangle$$
$$= D_{1q}\delta(t)\delta_{qq'}. \tag{3.1}$$

Then Eq. (2.13) would be modeled by

$$\frac{\partial P}{\partial t} + \sum_q \frac{\partial}{\partial h_{-q}} \left[(D_{0q} + D_{1q}) \frac{\partial}{\partial h_q} + \nu_q h_{-q} \right] P = 0, \tag{3.2}$$

an equation that is soluble, being a version of Hermite's equation. The full solution requires, of course, D_{1q}, but this can be done self-consistently. This is not the most general soluble form, for when problems in, say, plasma physics are resolved by the derivation of a Fokker Planck equation (see e.g., [5]) the form derived [for the velocity distribution $P(v)$ in that case] is

$$\frac{\partial P}{\partial t} + \sum_{i,j} \frac{\partial}{\partial v_i} D_{ij}(v) \frac{\partial}{\partial v_j} P + \sum_{i,j} \frac{\partial}{\partial v_i} \mu_{ij}(v) v_j P = 0, \tag{3.3}$$

where μ is called the dynamic friction. We use a somewhat similar form as a *starting point* for the derivation of a self-consistent model of Eq. (2.13),

$$\frac{\partial P}{\partial t} + \sum_q \frac{\partial}{\partial h_{-q}} \left[D_q \frac{\partial}{\partial h_q} + \omega_q h_{-q} \right] P \equiv \frac{\partial P}{\partial t} + L_0 P = 0. \tag{3.4}$$

Since this is a Hermite equation, it is soluble not only in the homogeneous form where

$$P_0 = N \exp\left(-\tfrac{1}{2} \sum h_q \phi_q^{-1} h_{-q} \right), \tag{3.5}$$

$$\frac{\partial P_0}{\partial t} = 0 \quad \text{with} \quad \phi_q = \frac{D_q}{\omega_q}, \tag{3.6}$$

but also as a Green function

$$\frac{\partial G}{\partial t} + L_0 G = \prod_q \delta(h_q - h_q') \delta(t-t'). \tag{3.7}$$

So

$$G=G(\cdots h_q h'_q h_{-q} h'_{-q},\ldots,t-t') \tag{3.8}$$

is available for expansions. In fact, although G is completely known, it turns out that only properties of the first few eigenfunctions will ever be needed, so we need not describe G in detail.

What we plan to do then is to study first the time-independent equation

$$LP=0 \tag{3.9}$$

by modeling its solution as an expansion around the solution of

$$L_0 P_0=0. \tag{3.10}$$

We have to find two equations for the functions D_q and ω_q that in some sense will make L_0 the best model of its kind describing L. It will turn out that the full details of one of the equations are not necessary and different candidates for that equation lead for small q to the same scaling relation relating ω_q and D_q (or ϕ_q). The scaling relation for the KPZ equation, which holds at all orders of our expansion, for small q, is

$$\omega_q^2 \alpha q^{d+4} \phi_q . \tag{3.11}$$

It is natural at this point to ask if problems of this kind have been studied in the literature before and the Fokker Planck equation for plasma [5], mentioned above, is such an example. Another is the nonlinear interaction of phonons with an interaction very much like that of the KPZ equation, but in a Schrödinger equation for the phonon field (instead of Fokker-Planck one in our case) [6]. That is a fundamentally simpler problem because the phonons are well approximated by the linear part of the equation except when they scatter, which comes from the nonlinear part. Peierls derived a Boltzmann equation governing the number of phonons n_k, which looks like

$$\frac{\partial n_k}{\partial t}+\int \kappa_1(k,j,l) n_k n_j d^3 j-\int \kappa_2(k,j,l) n_l n_j d^3 j=\text{sources}, \tag{3.12}$$

representing the annihilation of a phonon k when it meets a phonon j to form a phonon l $(l+k+j=0)$, with kernel K_1, and the creation of a k by the collision of j and l via K_2.

We expect to find, in analogy to this Peierls-Boltzmann (PB) equation, that the steady state of deposition is given by $\int \kappa_1(k,j,l)\phi_k\phi_j-\int \kappa_2(k,j,l)\phi_j\phi_l+\nu k^2\phi_k-D_k=0$, but unlike the PB equation where, as in the classic Boltzmann equation, the kernels κ_1 and κ_2 can be calculated from the linear parts of the equation, in the KPZ equation the kernels are themselves part of the calculation, for at low-k values the dynamics is dominated by the nonlinear integral terms and not by the $\nu\nabla^2 h$ and external noise parts. The actual kernels are quite reasonable functions expressed in terms of the characteristic q-dependent frequency ω_q. In terms of the ω's and the input D, our equation takes the form

$$\begin{aligned}\int & \frac{\widetilde{M}_{qjl}\widetilde{M}_{qjl}\phi_j\phi_l}{\omega_q+\omega_j+\omega_l}\,\delta(q+j+l)d^d j d^d l \\ &+\int \frac{\widetilde{M}_{qjl}\widetilde{M}_{jql}\phi_q\phi_l}{\omega_q+\omega_j+\omega_l}\,\delta(q+j+l)d^d j d^d l \\ &+\int \frac{\widetilde{M}_{qjl}\widetilde{M}_{ljq}\phi_q\phi_j}{\omega_q+\omega_j+\omega_l}\,\delta(q+j+l)d^d j d^d l+D_{0q}-\nu_q\phi_q \\ &=0,\end{aligned} \tag{3.13}$$

where

$$\widetilde{M}_{qlm}=\sqrt{\frac{2\Omega}{(2\pi)^d}}\,M_{qlm} \tag{3.14}$$

and ω is obtained in Sec. V. The two terms with ϕ_q are equal, but it is so written to show the symmetry. The equation for ω is derived in Sec. V below and is of a similar level of complexity. There is some difference in the signs compared to the Peierls-Boltzmann equation. In the PB equation the two nonlinear terms are positive and a natural minus appears between them as in the classic Boltzmann equation. In our equation (3.13) the term $\widetilde{M}_{qjl}^2$ is clearly positive, but the signs of the other terms such as $\widetilde{M}_{qjl}\widetilde{M}_{jql}$ depend on q, j, and l.

To derive Eq. (3.13) we write Eq. (3.9) as

$$\begin{aligned}LP\equiv\sum_q \frac{\partial}{\partial h_{-q}}\Bigg[& D_q\frac{\partial}{\partial h_q}+\omega_q h_{-q}+\sum_{l,m} M_{qlm}h_l h_m \\ &+(D_{0q}-D_q)\frac{\partial}{\partial h_q}+(\nu_q-\omega_q)h_{-q}\Bigg]P=0\end{aligned} \tag{3.15}$$

and expand P about P_0 defined by

$$L_0 P_0\equiv\sum_q \frac{\partial}{\partial h_{-q}}\left[D_q\frac{\partial}{\partial h_q}+\omega_q h_{-q}\right]P_0=0, \tag{3.16}$$

i.e.,

$$P_0=N\exp\left[-\tfrac{1}{2}\sum h_q h_{-q}/\phi_q\right], \tag{3.17}$$

where N is the normalization yielding $\int P_0=1$ and

$$\int h_q h_{-q} P_0\{h\} Dh=\phi_q . \tag{3.18}$$

Readers who are not interested in the technical derivation of Eq. (3.13) and in the derivation of the equation for ω in Eq. (5.4) but only in their solution can jump to Sec. VI, where the power-law solutions are derived, basically by showing that the power law for ϕ_q and consequently for ω_q will solve Eq. (3.13) for small q.

IV. EXPANSION OF THE TWO-POINT FUNCTION

We are interested in average quantities such as the two-point function

$$\langle h_q h_{-q}\rangle = \int h_q h_{-q} P\ Dh. \tag{4.1}$$

In general, Eq. (2.15) can be used to obtain a hierarchy of equations relating n-point functions to $(n+1)$-point functions. This hierarchy will be used to derive Eq. (3.13). The basic step in obtaining the hierarchy is to note that by multiplying Eq. (2.15) by some function of the fields F that is well behaved and integrating by parts we obtain the general relation

$$\sum_P D_{0p}\left\langle \frac{\partial^2 F}{\partial h_p \partial_{h-p}}\right\rangle - \nu_p \left\langle h_{-p} \frac{\partial F}{\partial h_{-p}}\right\rangle - \sum_{P,l,m} M_{Plm}\left\langle h_l h_m \frac{\partial F}{\partial h_{-p}}\right\rangle = 0. \tag{4.2}$$

Thus, if we choose $F=\frac{1}{2}h_q h_{-q}$, we obtain

$$D_{0q} - \nu_q \langle h_q h_{-q}\rangle - \sum_{l,m} M_{qlm}\langle h_l h_m h_q\rangle = 0, \tag{4.3}$$

and if we take $F=h_l h_m h_q$ with indices such that the sum of any two is not zero, we obtain

$$-[\nu_q+\nu_l+\nu_m]\langle h_l h_m h_q\rangle - \sum_{l',m'} [M_{-ll'm'}\langle h_{l'} h_{m'} h_m h_q\rangle + M_{-ml'm'}\langle h_l h_{m'} h_l h_q\rangle + M_{-ql'm'}\langle h_{l'} h_{m'} h_l h_m\rangle] = 0, \tag{4.4}$$

and this goes on.

This hierarchy is exact and we are interested in obtaining, say, the two-point function in an expansion around L_0. If we write

$$P = P_0 + P_1 + P_2 + \cdots \tag{4.5}$$

and ascribe to P_1 the order M, to P_2 the order M^2, and so on, and within Eq. (3.15) we ascribe order M^2 to $(D_0 - D)$ and $(\nu-\omega)$, we obtain the following equations in schematic form to determine P_1, P_2, etc.

$$L_0 P_0 = 0, \tag{4.6}$$

$$L_0 P_1 = -\frac{\partial}{\partial h} MhhP_0, \tag{4.7}$$

$$L_0 P_2 = -\frac{\partial}{\partial h} MhhP_1 - \frac{\partial}{\partial h}(D_0 - D)\frac{\partial P_0}{\partial h} - \frac{\partial}{\partial h}(\nu-\omega)hP_0, \tag{4.8}$$

etc. [The way we attached different orders of M to different parts of $L-L_0$ may look arbitrary. In Ref. [2] we attached to all the parts of $L-L_0$ order M. The equation obtained there is slightly different from Eq. (3.13), but that difference has no bearing on the small-q behavior of the two-point function, as will become evident later.] The expansion of the probability distribution P implies a corresponding expansion for any average

$$\langle F\rangle = \langle F\rangle_0 + \langle F\rangle_1 + \langle F\rangle_2 + \cdots, \tag{4.9}$$

where $\langle F\rangle_i$ is $\int FP_i Dh$. We also denote

$$\langle F\rangle^{(n)} = \sum_{i=0}^{n} \langle F\rangle_i. \tag{4.10}$$

Consider first Eq. (4.4). We are interested in $\langle h_l h_m h_q\rangle^{(1)}$. It is obtained from

$$\begin{aligned}&[\omega_l+\omega_m+\omega_q]\langle h_l h_m h_q\rangle^{(1)}\\ &-[\nu_l+\nu_m+\nu_q-\omega_l-\omega_m-\omega_q]\langle h_l h_m h_q\rangle^{(0)}\\ &-\sum_{l',m'}[M_{-ll'm'}\langle h_{l'} h_{m'} h_m h_q\rangle^{(0)}\\ &+M_{-ml'm'}\langle h_{l'} h_{m'} h_l h_q\rangle^{(0)}\\ &+[M_{-ql'm'}\langle h_{l'} h_{m'} h_l h_m\rangle^{(0)}] = 0.\end{aligned} \tag{4.11}$$

By definition of P_0, $\langle h_l h_m h_q\rangle^{(0)}=0$ and the required four-point functions $\langle hhhh\rangle^{(0)}$ are easily calculated.

The final result is

$$\langle h_l h_m h_q\rangle^{(1)} = -2[M_{lmq}\phi_m\phi_q + M_{mlq}\phi_l\phi_q + M_{qlm}\phi_l\phi_m]/[\omega_l+\omega_m+\omega_q]. \tag{4.12}$$

[In Eq. (4.12) the symmetry in the two last indices of M and the reflection symmetry in the indices is used.] Equation (4.3), when taken to first order, yields

$$D_q - \omega_q\langle h_q h_q\rangle_1 - \sum M_{qlm}\langle h_q h_l h_m\rangle_0 = 0, \tag{4.13}$$

i.e.,

$$\langle h_q h_q\rangle^{(1)} = \phi_q. \tag{4.14}$$

The same equation considered to second order gives

$$D_q - \omega_q\langle h_q h_{-q}\rangle^{(2)} + (D_{0q} - D_q) + (\omega_q - \nu_q)\langle h_q h_{-q}\rangle^{(1)} + \sum_{l,m} M_{qlm}\langle h_l h_m h_q\rangle^{(1)} = 0. \tag{4.15}$$

We require now that within our approximation $\langle h_q h_{-q}\rangle = \phi_q$, namely, $\langle h_q - h_{-q}\rangle^{(2)} = \phi_q$. Using the expression obtained for $\langle h_q h_{-q}\rangle^{(1)}$ and $\langle h_l h_m h_q\rangle^{(1)}$, Eq. (3.13) is obtained.

It is interesting to consider also the expansion for the distribution function P. In order to obtain a structure familiar from quantum field theory, we apply a similarity transformation to all the operators h_q and $\Pi_q = \partial/\partial h_q$. The transformed operator $\widetilde{A}$ is given in terms of the original operator A by

$$\widetilde{A} = \exp\left[\frac{1}{4}\sum \frac{h_p h_{-p}}{\phi_p}\right] A \exp\left[-\frac{1}{4}\sum \frac{h_p h_{-p}}{\phi_p}\right], \tag{4.16}$$

i.e.,

$$\widetilde{h}_q = h_q \tag{4.17}$$

and

$$\tilde{\Pi}_q=\Pi_q-\frac{1}{2}\frac{h_{-q}}{\phi_q}. \tag{4.18}$$

We define now ''excitation'' creation and destruction operators $\eta_q^\dagger$ and η_q

$$\eta_q^\dagger=\frac{1}{2\sqrt{\phi_q}}h_{-q}-\sqrt{\phi_q}\Pi_q, \tag{4.19}$$

$$\eta_q=\frac{1}{2\sqrt{\phi_q}}h_q+\sqrt{\phi_q}\Pi_{-q}. \tag{4.20}$$

In terms of these operators, which obey the usual Bose commutation relations $[\eta_q,\eta_p]=[\eta_q^\dagger,\eta_p^\dagger]=0$ and $[\eta_q,\eta_p^\dagger]=\delta_{qp}$, the equation for steady state $\partial P/\partial t=0$ is transformed into

$$\left\{\sum_q \omega_q\eta_q^\dagger\eta_q+\sum_{q,l,m}M_{qlm}\frac{\sqrt{\phi_l}\sqrt{\phi_m}}{\sqrt{\phi_q}}\eta_{-q}^\dagger \times(\eta_l+\eta_{-l}^\dagger)(\eta_m+\eta_{-m}^\dagger)-\sum_q(D_{0q}-D_q)\times\frac{1}{\phi_q}\eta_q^\dagger\eta_{-q}^\dagger+\sum_q(\nu_q-\omega_q)\eta_q^\dagger(\eta_q+\eta_{-q}^\dagger)\right\}|S\rangle=0, \tag{4.21}$$

where $|S\rangle$ is the ''true ground state.'' Once $|S\rangle$ is obtained, a steady-state average of any functional A of the h_q's is given by

$$\int PA\{h_q\}Dh=\langle 0|A\{h_q\}|S\rangle, \tag{4.22}$$

where $|0\rangle$ is the vacuum state defined by $\eta_q|0\rangle=0$ for all q. The function $|S\rangle$ is expanded now in an expansion corresponding to the expansion of P,

$$|S\rangle=|S_0\rangle+|S_1\rangle+|S_2\rangle+\cdots. \tag{4.23}$$

Clearly,

$$|S_0\rangle=|0\rangle \tag{4.24}$$

and

$$|S_1\rangle=\sum\frac{M_{qlm}\sqrt{\phi_l}\sqrt{\phi_m}}{[\omega_q+\omega_l+\omega_m]\sqrt{\phi_q}}\eta_q^\dagger\eta_l^\dagger\eta_m^\dagger|0\rangle. \tag{4.25}$$

$|S_0\rangle$ is the ''unperturbed ground state'' and $|S_1\rangle$ is a function summing with the appropriate weights the states where the modes q, l, and m have gone into the first excited state. [In deriving Eq. (4.25) two properties of M_{qlm} that hold for the KPZ equation were used: $M_{qlm}=M_{-q,-l,-m}$ and $M_{qlm}=0$ if any of the indices vanish, recall $q+l+m=0$.]

The equation for $|S_2\rangle$ is

$$\sum_q\omega_q\eta_q^\dagger\eta_q|S_2\rangle+\sum_{q,l,m}M_{qlm}\frac{\sqrt{\phi_l}\sqrt{\phi_m}}{\sqrt{\phi_q}}\eta_{-q}^\dagger(\eta_l+\eta_{-l}^\dagger)\times(\eta_m+\eta_{-m}^\dagger)|S_1\rangle-\left\{\sum_q(D_{0q}-D_q)\frac{1}{\phi_q}\eta_q^\dagger\eta_{-q}^\dagger+\sum_q(\nu_q-\omega_q)\eta_q^\dagger(\eta_q+\eta_{-q}^\dagger)\right\}|0\rangle=0. \tag{4.26}$$

$|S_2\rangle$ has the contribution of states with two, four, and six excitations. Of special interest are the states of two excitations. Clearly, the momenta of the two must be paired. Denoting that part by $|S_2\rangle^{(2)}$, we find

$$|S_2\rangle^{(2)}=\left\{\sum_q\left(D_{0q}\frac{1}{\phi_q}-\nu_q\right)\Big/2\omega q+2\sum M_{qlm}\left(M_{qlm}\frac{\phi_l\phi_m}{\phi_q}+M_{lqm}\phi_m+M_{mql}\phi_l\right)\Big/[2\omega_q(\omega_q+\omega_l+\omega_m)]\right\}\eta_q^\dagger\eta_{-q}^\dagger|0\rangle. \tag{4.27}$$

Equation (3.13), which ensures that the correlation $\langle h_q h_q\rangle$ calculated to second order is identical to ϕ_q, implies at once that $|S_2\rangle^{(2)}$ vanishes identically.

Equation (3.13) involves two unknown functions ϕ_q and ω_q. In order to obtain ϕ_q we need another equation relating the two.

V. DYNAMIC ARGUMENTS

We suggest that ϕ_q and ω_q must be chosen in such a way that L_0 is the best model of its kind describing the true time evolution operator L. We have already made one choice by deciding that $\langle h_q h_q\rangle$ calculated to second order be identical to ϕ_q. The other natural choice involves ω_q. Since L_0 is a very simple evolution operator, it is easily verified that ω_q is the inverse lifetime associated with the mode q. In technical terms ω_q as a function of q is the single excitation spectrum of the evolution operator $-\tilde{L}_0$, obtained by the similarity transformation (4.16) from $-L_0$, namely, $\Sigma\omega_q\eta_q^\dagger\eta_q$. One possible way of fitting $\tilde{L}_0$ to $\tilde{L}$ is to identify ω_q with the value of its perturbed counterpart. (We will see later that there are many ways of obtaining an equation for ω_q, but certainly fitting the ''single excitation'' spectra of $\tilde{L}_0$ and $\tilde{L}$ seems to be a reasonable procedure.) Let $|\psi_q\rangle$ be the eigenstate of $\tilde{L}$ with momentum q that is obtained perturbatively from $|\psi_q\rangle^{(0)}=\eta_q^\dagger|0\rangle$. We expand

$$|\psi_q\rangle=|\psi_q\rangle^{(0)}+|\psi_q\rangle^{(1)}+|\psi_q\rangle^{(2)}+\cdots \tag{5.1}$$

and the associated eigenvalue $\tilde{\omega}_q$ is expanded

$$\tilde{\omega}_q=\tilde{\omega}_q^{(0)}+\tilde{\omega}_q^{(1)}+\tilde{\omega}_q^{(2)}+\cdots, \tag{5.2}$$

where, clearly, $\tilde{\omega}_q^{(0)}=\omega_q$. The calculation of $\tilde{\omega}_q^{(1)}$ and $\tilde{\omega}_q^{(2)}$ is a straightforward perturbation expansion that takes into account that some terms in $\tilde{L}$ are order M and some are order

M^2. The calculation is quite similar to that of the true ground state $|S\rangle$ presented in Sec. IV.

We find

$$\tilde{\omega}_q^{(1)}=0 \tag{5.3}$$

and

$$\tilde{\omega}_q^{(2)}=\nu_q-\omega_q-2\sum M_{qlm}\frac{M_{lmq}\phi_m+M_{mlq}\phi_l}{\omega_l+\omega_m}, \tag{5.4}$$

where as in the derivation of Eq. (4.25) we have used the fact that $M_{ql,-l}=0$ and some symmetry properties of the kernel M_{qlm} that hold for the KPZ equation (e.g., $M_{qlm}=M_{-q,-l,-m}$ and $M_{qlm}=M_{qml}$). Equation (5.4) was originally derived by Herring in a different context [7]. Note that expression (3.13) contains three ω's and (5.4) two. If we require that the eigenvalue ω_q is unchanged, we obtain as a second equation relating ϕ and ω

$$\nu_q-\omega_q-2\sum M_{qlm}\frac{M_{lmq}\phi_m+M_{mlq}\phi_l}{\omega_l+\omega_m}=0. \tag{5.5}$$

Equation (5.5) is based on the requirement that the model L_0 should be fitted to the real evolution operator by fitting the ''excitation spectrum'' of the ''perturbed singly excited states.'' In the following, we describe the different ''characteristic frequency'' fitting of the model, which was used in Ref. [2].

Consider a system in steady state. We can measure at time zero an observable A and after time t has elapsed we can measure another observable B. This leads to the definition of a correlation

$$\langle A(0)B(t)\rangle=\int P\{h_q^{(1)}\}A\{h_q^{(1)}\}P\{h_q^{(1)},h_q^{(2)},t\}\times B\{h_q^{(2)}\}Dh_q^{(1)}Dh_q^{(2)}, \tag{5.6}$$

where $P\{h_q^{(1)}\}$ is the steady-state distribution and $P\{h_q^{(1)},h_q^{(2)},t\}$ is the distribution of the $h_q^{(2)}$'s obtained after time t when the initial condition at time zero is $\Pi\,\delta(h_q^{(2)}-h_q^{(1)})$. The characteristic frequency associated with h_q is customarily defined by

$$\frac{1}{\overline{\omega_q}}=\frac{\int_0^\infty\langle h_q(0)h_{-q}(t)\rangle dt}{\langle h_qh_{-q}\rangle_S}, \tag{5.7}$$

where in the denominator the subscript S denotes steady-state averaging.

Using the equation of motion (2.13) for $P\{h_q^{(1)},h_q^{(2)},t\}$, multiplying by $P\{h_q^{(1)}\}A\{h_q^{(1)}\}B\{h_q^{(2)}\}$, integrating over $h_q^{(1)}$, and integrating by parts over $h_q^{(2)}$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\langle A(0)B(t)\rangle=&\sum_q D_{0q}\left\langle A(0)\frac{\partial^2B(t)}{\partial h_q\partial h_{-q}}\right\rangle\\&-\sum_q\nu_q\left\langle A(0)h_{-q}\frac{\partial B(t)}{\partial h_{-q}}\right\rangle\\&-\sum M_{qlm}\left\langle A(0)h_lh_m\frac{\partial B(t)}{\partial h_{-q}}\right\rangle\\&+\delta(t)\langle AB\rangle_S,\end{aligned} \tag{5.8}$$

where the last term on the right-hand side arises due to the arbitrary choice $\langle A(0)B(t)\rangle=0$ for $t<0$, i.e., it serves to introduce the initial conditions. Since the expression for ω_q introduced in Eq. (5.7) involves integration from zero to infinity, which can be replaced by integration from $-\infty$ to ∞ [recall that we chose $\langle h_q(0)h_{-q}(t)\rangle=0$ for $t<0$], we use the integrated version of Eq. (5.8),

$$\begin{aligned}&\sum_q D_{0q}\int_{-\infty}^{\infty}dt\left\langle A(0)\frac{\partial^2B(t)}{\partial h_q\partial h_{-q}}\right\rangle\\&-\sum_q\nu_q\int_{-\infty}^{\infty}dt\left\langle A(0)h_{-q}(t)\frac{\partial B(t)}{\partial h_{-q}}\right\rangle\\&-\sum M_{qlm}\int_{-\infty}^{\infty}dt\left\langle A(0)h_l(t)h_m(t)\frac{\partial B}{\partial h_{-q}}(t)\right\rangle\\&+\langle AB\rangle_S=0,\end{aligned} \tag{5.9}$$

where the term arising from the time derivative is zero because $\langle A(0)B(t)\rangle$ is zero for negative time and must tend to zero for positive t tending to infinity.

We expand

$$\frac{1}{\overline{\omega_q}}=\left(\frac{1}{\overline{\omega_q}}\right)^{(0)}+\left(\frac{1}{\overline{\omega_q}}\right)^{(1)}+\left(\frac{1}{\overline{\omega_q}}\right)^{(2)}+\cdots, \tag{5.10}$$

where clearly $(1/\overline{\omega_q})^{(0)}=1/\omega_q$. Choosing $A=h_q$ and $B=h_{-q}$, it is easily verified that

$$\left(\frac{1}{\overline{\omega_q}}\right)^{(1)}=0. \tag{5.11}$$

Second-order calculation yields

$$-\omega_q\left(\frac{1}{\overline{\omega_q}}\right)^{(2)}-\sum\frac{M_{qlm}}{\phi_q}\int_{-\infty}^{\infty}\langle h_q(0)h_l(t)h_m(t)\rangle^{(1)}dt-(\nu_q-\omega_q)\left(\frac{1}{\overline{\omega_q}}\right)^{(0)}=0. \tag{5.12}$$

The next step is to obtain $\int_{-\infty}^{\infty}\langle h_q(0)h_l(t)h_m(t)\rangle^{(1)}$. This is easy. Equation (5.9) is to be used now with $B=h_lh_m$ and $A=h_q$,

$$-[\omega_l+\omega_m]\int_{-\infty}^{\infty}dt\langle h_q(0)h_l(t)h_m(t)\rangle^{(1)}$$
$$-\sum M_{-lkj}\int_{-\infty}^{\infty}dt\langle h_q(0)h_k(t)h_j(t)h_m(t)\rangle^{(0)}$$
$$-\sum M_{-mkj}\int_{-\infty}^{\infty}\langle h_q(0)h_k(t)h_j(t)h_l(t)\rangle^{(0)}$$
$$+\langle h_q h_l h_m\rangle_S^{(1)}=0. \tag{5.13}$$

The expression for $\int_{-\infty}^{\infty}dt\langle h_q(0)h_k(t)h_j(t)h_l(t)\rangle^{(0)}$ is very simple to calculate and the final equation obtained by demanding in Eq. (5.12) that $(1/\omega_q)^{(2)}$ vanishes is

$$\omega_q-\nu_q+2\sum\frac{M_{qlm}[M_{lmq}\phi_m+M_{mlq}\phi_l]}{\omega_l+\omega_m}+2\omega_q\sum_{l,m}\times\frac{M_{qlm}[M_{lmq}\phi_m+M_{mlq}\phi_l+M_{qlm}\phi_l\phi_m/\phi_q]}{[\omega_l+\omega_m][\omega_l+\omega_m+\omega_q]}=0. \tag{5.14}$$

We see that the different fittings of the model L_0 to the real evolution equation lead to two different equations that can serve as candidates for a second equation, relating ω_q and ϕ_q. In fact, one could think of many criteria for fitting L_0 to L, which will lead of course to equations different from Eq. (5.5) or (5.14). This may seem alarming, but L_0 is not to replace L but just should serve as a reasonable simple evolution operator to expand about. More importantly we show in the Appendix that these details or even the difference introduced by ascribing say, order M to all terms in $L-L_0$ are irrelevant. The small-q behavior of ϕ_q and ω_q is totally unaffected by these differences.

It will be noted that the series we develop is not that of the Feynman diagrams. The reason is that instead of the Dyson "Σ," we have the input and output kernels, whose pattern appears at each order and is vital to the power-law solution. It is possible to derive diagrammatics analogous to the Feynman diagrams, the details of which are in Refs. [8–10].

VI. ASYMPTOTIC POWER-LAW SOLUTION

In the following we will treat the specific equations (3.13) and (5.4). This treatment will ensure, however, that if we started, for example, from Eqs. (3.13) and (5.14) the obtained small-q behavior would not have changed. Equations (3.13) and (5.5) can be written in the form

$$D_0-\nu q^2\phi_q-I_1(q)\phi_q+I_2(q)=0 \tag{6.1}$$

and

$$\omega_q-\nu q^2-J(q)=0, \tag{6.2}$$

where

$$I_1(q)=\frac{2g^2}{(2\pi)^d}\int d^dl\,\frac{\vec{l}\cdot(\vec{q}-\vec{l})}{\omega_l+\omega_{q-l}+\omega_q}\times[\vec{l}\cdot\vec{q}\phi_l+(\vec{q}-\vec{l})\cdot\vec{q}\phi_{q-l}], \tag{6.3}$$

$$I_2(q)=\frac{2g^2}{(2\pi)^d}\int d^dl\,\frac{[\vec{l}\cdot(\vec{q}-\vec{l})]^2}{\omega_l+\omega_{q-l}+\omega_q}\phi_l\phi_{q-l}, \tag{6.4}$$

and

$$J(q)=\frac{2g^2}{(2\pi)^d}\int d^dl\,\frac{\vec{l}\cdot(\vec{q}-\vec{l})}{\omega_l+\omega_{q-l}}[\vec{l}\cdot\vec{q}\phi_l+(\vec{q}-\vec{l})\cdot\vec{q}\phi_{q-l}]. \tag{6.5}$$

We expect that for small enough q, ϕ_q and ω_q are power laws in q. Namely, there exists a q_0 small enough such that for all $|q|<q_0$, ϕ_q and ω_q are adequately described by

$$\phi_q=Aq^{-\Gamma} \tag{6.6}$$

and

$$\omega_q=Bq^{\mu}. \tag{6.7}$$

We are interested in Eqs. (6.1) and (6.2) for small q's only. The difficulty, however, is that the integrals involved, $I_1(q)$, $I_2(q)$, and $J(q)$, have contributions from large l's as well as from small l's. Our first task thus is to separate the low momenta from the high momenta in the coupled integral equations (6.1) and (6.2). We break up each of the integrals $I(q)$ and $J(q)$ into the sum of two contributions $I^{>}(q)$,$J^{>}(q)$ and $I^{<}(q)$,$J^{<}(q)$, corresponding to domains of integration with $|l|>q_0$ and $|l|<q_0$, respectively. By expanding with respect to q and ω_q we obtain the leading small-q behavior of the $I^{>}(q)$ and $J^{>}(q)$,

$$I_1^{>}(q)=A_1q^2+B_1\omega_q q^2+C_1q^4, \tag{6.8}$$

$$I_2^{>}(q)=A_2+B_2\omega_q+C_2q^2, \tag{6.9}$$

and

$$J^{>}(q)=A_3q^2. \tag{6.10}$$

Retaining only the leading terms, Eqs. (6.1) and (6.2) reduce now to

$$D_0+A_2-(\nu+A_1)q^2\phi_q-I_1^{<}(q)\phi_q+I_2^{<}(q)=0 \tag{6.11}$$

and

$$\omega_q-(\nu+A_3)q^2-J^{<}(q)=0. \tag{6.12}$$

The advantage of Eqs. (6.11) and (6.12) over Eqs. (6.1) and (6.2) is that at the mere price of renormalizing some constants in both equations, we are left with the integrals $I^{<}(q)$ and $J^{<}(q)$ that can be calculated explicitly given the simple power laws (6.6) and (6.7) for ϕ_l and ω_l that hold for $|l|<q_0$. In particular, each of these integrals is a power law in q with an exponent that depends only on Γ, μ, and the dimension d. The treatment of Eqs. (6.11) and (6.12) is carried on by studying the various possibilities of different terms to dominate the equations at small q.

The full details of that investigation are given in the Appendix. Here we just quote the final results. We find two consistent possibilities. For $d>2$ we find the possibility $\Gamma=\mu=2$. This corresponds to a weak-coupling solution, which is also obtainable by a conventional expansion in M. The second option is to have a strong-coupling solution that is characterized by the scaling relation [11]

$$d+4-\Gamma-2\mu=0 \tag{6.13}$$

and the additional equation that fixes the exponents $F(\Gamma,\mu)=0$, where the function F is given by

$$\begin{aligned} F(\Gamma,\mu)=&-\int d^d t\,\frac{\vec{t}\cdot(\hat{e}-\vec{t})}{[t^{\mu}+|\hat{e}-\vec{t}|^{\mu}+1]} \\ &\times[\vec{t}\cdot\hat{e}t^{-\Gamma}+(\hat{e}-\vec{t})\cdot\hat{e}|\hat{e}-\vec{t}|^{-\Gamma}] \\ &+\int d^d t\,\frac{[\vec{t}\cdot(\hat{e}-\vec{t})]^2}{[t^{\mu}+|\hat{e}-\vec{t}|^{\mu}+1]}\,t^{-\Gamma}|\hat{e}-\vec{t}|^{-\Gamma}, \end{aligned} \tag{6.14}$$

where $\hat{e}$ is an arbitrary fixed unit vector and the integration is over the whole $\vec{t}$ space.

In one dimension it may be easily verified by direct inspection that $F(2,\mu)=0$, regardless of the value of μ. So we recover the exact result $\Gamma=2$ and from the scaling relation we obtain $\mu=3/2$.

In $d=2$ we obtain $\mu(\Gamma)$ from the scaling relation and solve numerically the equation

$$F(\Gamma,\mu(\Gamma))=0. \tag{6.15}$$

The solution is $\Gamma=2.59$, which is in good agreement with results obtained by numerical simulations [12–17]. The results of Bouchaud and Cates [18] and of Perlsman and Schwartz [19] are also similar.

VII. HIGHER-ORDER EXPANSIONS

The results obtained so far are based on fitting a model L_0 to describe the true evolution operator L, by expanding in the difference between L and L_0 to second order. It is important, however, to try to understand the behavior of terms obtained from higher-order expansions. The main reason is to check whether higher-order corrections may render the expansion inconsistent in an obvious way and to see whether more information can be extracted by such corrections. We start by considering the weak-coupling solution, i.e., consider the case $d>2$, $\Gamma=2$, and $\mu=2$. In that case $I_1^<(q)$ is proportional to q^2, $I_2^<(q)$ is a constant to leading order in q, and $J(q)$ is proportional to q^2. A direct check of Eqs. (6.11) and (6.12) indicates that indeed the solution is $\Gamma=\mu=2$.

For the strong-coupling case we find that Eq. (6.11) becomes

$$\begin{aligned} D_0+A_2-A(\nu+A_1)q^{2-\Gamma}& \\ +2\frac{A^2}{B}\frac{g^2}{(2\pi)^d}F(\Gamma,\mu)q^{d+4-2\Gamma-\mu}&=0 \end{aligned} \tag{7.1}$$

and Eq. (6.12) reduces to

$$Bq^{\mu}-(\nu+A_3)q^2-\frac{2Ag^2}{B(2\pi)^d}G(\Gamma,\mu)q^{d+4-2\Gamma-\mu}=0, \tag{7.2}$$

where $G(\Gamma,\mu)$ is given by

$$\begin{aligned} G(\Gamma,\mu)=&\int d^d t\,\frac{\vec{t}\cdot(\hat{e}-\vec{t})}{[t^{\mu}+|\hat{e}-\vec{t}|^{\mu}]} \\ &\times[\vec{t}\cdot\hat{e}t^{-\Gamma}+(\hat{e}-\vec{t})\cdot\hat{e}|\hat{e}-\vec{t}|^{-\Gamma}]. \end{aligned} \tag{7.3}$$

As detailed in the Appendix, the two dominant terms in Eq. (7.2) are the first and last on the left-hand side of the equation. Equation (7.1), on the other hand, has only a single dominant term, which is the last one on the left-hand side of the equation. Therefore, the coefficient of that dominant power of $q,F(\Gamma,\mu)$ must vanish.

When going to higher-order terms in the same expansion, we see that the terms that produce dominant terms in the $2n$th order, involve n d-dimensional integrations, $2n$ M's, $(n+1)$ ϕ's, and $(2n-1)$ ω denominators. Denoting that power of q by α_n and $\theta=d+4-\Gamma-2\mu$ we obtain

$$\alpha_n=\begin{cases} d+4-2r-\mu+(n-1)\theta & \text{if } d+4-2\Gamma-\mu+(n-1)\theta<0 \\ 0 & \text{if } d+4-2\Gamma-\mu+(n-1)\theta\geqslant 0 \end{cases}. \tag{7.4}$$

For the weak-coupling solution ($d>2$, $\Gamma=2$, and $\mu=2$), the second possibility in Eq. (7.4) is encountered since $\theta>0$. Therefore, the introduction of higher-order terms does not change the fact that the weak-coupling solution remains consistent.

The strong-coupling solution is more interesting. If we had $\theta<0$, each higher-order correction would be more important than the previous one. This would render the whole expansion meaningless, as the underlying expectation that higher-order terms are less important is violated. If, on the other hand, θ were larger than zero, higher-order terms are less important, as expected, and the second-order terms in Eqs. (7.1) and (7.2) would already give the exact result. This cannot happen, however, since Eq. (7.2) then determines the scaling relation that implies $\theta=0$. We see that the scaling relation $\theta=0$ plays a very important role when higher-order terms are considered. It controls the expansion in such a way that the most dominant q dependences arising from different orders of the expansion are *identical*.

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APPENDIX: DETAILED ASYMPTOTIC SOLUTION

In this appendix we study the possible solutions of Eqs. (6.11) and (6.12), namely, the possible values of Γ and μ. First we consider the q dependence of the integral $I_1^<(q)$, $I_2^<(q)$, and $J^<(q)$. We find by transforming the variable of integration from $\vec{l}$ to $\vec{t}=\vec{l}/|q|$,

$$I_1^<(q)=\frac{2g^2}{(2\pi)^d}\frac{A}{B}q^{d+4-\Gamma-\mu}\int^{q_0/q}d^dt\,\frac{\vec{t}\cdot(\hat{e}-\vec{t})}{t^\mu+|\hat{e}-\vec{t}|^\mu+1}\times[\vec{t}\cdot\hat{e}t^{-\Gamma}+(\hat{e}-\vec{t})\cdot\hat{e}|\vec{t}-\hat{e}|^{-\Gamma}], \tag{A1}$$

$$I_2^<(q)=\frac{2g^2}{(2\pi)^d}\frac{A^2}{B}q^{d+4-2\Gamma-\mu}\int^{q_0/q}d^dt\times\frac{[\vec{t}\cdot(\hat{e}-\vec{t})]^2}{t^\mu+|\hat{e}-\vec{t}|^\mu+1}t^{-\Gamma}|\hat{e}-\vec{t}|^{-\Gamma}, \tag{A2}$$

and

$$J^<(q)=\frac{2g^2}{(2\pi)^d}\frac{A}{B}q^{d+4-\Gamma-\mu}\int^{q_0/q}d^dt\,\frac{\vec{t}\cdot(\hat{e}-\vec{t})}{t^\mu+|\hat{e}-\vec{t}|^\mu}\times[\vec{t}\cdot\hat{e}t^{-\Gamma}+(\hat{e}-\vec{t})\cdot\hat{e}|\hat{e}-\vec{t}|^{-\Gamma}], \tag{A3}$$

where $\int^{q_0/q}$ means that $|\vec{t}|$ is restricted below q_0/q and $\hat{e}$ is an arbitrary fixed unit vector. The small-q dependence of each of the integrals depends on whether or not it converges when $|\vec{t}|$ is not restricted at all. We obtain

$$I_1^<(q),J^<(q)\propto\begin{cases} q^2 & \text{for } d+2-\Gamma-\mu>0\\ q^2\ln\dfrac{q_0}{q} & \text{for } d+2-\Gamma-\mu=0\\ q^{d+4-\Gamma-\mu} & \text{for } d+2-\Gamma-\mu<0\end{cases} \tag{A4}$$

and

$$I_2^<(q)\propto\begin{cases} \text{const} & \text{for } d+4-2\Gamma-\mu>0\\ \text{const}\times\ln\dfrac{q_0}{q} & \text{for } d+4-2\Gamma-\mu=0\\ q^{d+4-2\Gamma-\mu} & \text{for } d+4-2\Gamma-\mu<0.\end{cases} \tag{A5}$$

We consider now the upper-right quadrant of the (Γ,μ) plane, where a solution may be expected. The lines $d+2-\Gamma-\mu=0$ and $d+4-2\Gamma-\mu=0$ divide the quadrant into four sectors. We investigate next each sector separately to decide whether or not a solution can exist there. Section α is defined by $d+2-\Gamma-\mu>0$ and $d+4-2\Gamma-\mu>0$. In that sector Eqs. (6.11) and (6.12) reduce to

$$D_0+A_2+A_2'-(\nu+A_1+A_1')Aq^{2-\Gamma}=0 \tag{A6}$$

and

$$Bq^\mu-(\nu+A_3+A_3')q^2=0. \tag{A7}$$

The conclusion is $\Gamma=2$ and $\mu=2$. By definition of the sectors it follows that this can happen only for $d>2$. In section β, $d+2-\Gamma-\mu>0$ and $d+4-2\Gamma-\mu<0$. In this sector Eq. (A6) is replaced by

$$D_0+A_2-(\nu+A_1+A_1')Aq^{2-\Gamma}+C_2q^{d+4-2\Gamma-\mu}=0. \tag{A8}$$

The last term on the left-hand side of Eq. (A8) is negligible compared to the second term because of the defining condition $d+2-\Gamma-\mu>0$. It follows that $\Gamma=2$. This implies the contradiction $\mu>d$ and $\mu<d$ and at the same time Eq. (A7) yields also $\mu=2$. (Points on the boundaries of the sectors will be discussed separately later.) Section γ is the section where $d+2-\Gamma-\mu<0$ and $d+4-2\Gamma-\mu>0$. Equation (A6) is replaced by

$$D_0+A_2-(\nu+A_1)Aq^{2-\Gamma}+C'q^{d+4-2\Gamma-\mu}=0. \tag{A9}$$

The two equations defining the sector imply that the dominant term on the left-hand side of the equation is the constant D_0+A_2. (The alternative is $\Gamma=2$, which leads to the contradiction $\mu>d$ and $\mu<d$.) A necessary condition for the existence of a solution is $D_0+A_2=0$, but this is impossible because D_0 and A_2 are both positive definite.

Section δ is defined by $d+2-\Gamma-\mu<0$ and $d+4-2\Gamma-\mu<0$. In this sector Eqs. (6.11) and (6.12) take the form

$$D_0+A_2-(\nu+A_1)Aq^{2-\Gamma}+\frac{2g^2}{(2\pi)^d}\frac{A^2}{B}F(\Gamma,\mu)^{d+4-2\Gamma-\mu}=0 \tag{A10}$$

and

$$Bq^\mu-(\nu+A_3)q^2-\frac{2g^2}{(2\pi)^d}\frac{A}{B}G(\Gamma,\mu)q^{d+4-2\Gamma-\mu}=0, \tag{A11}$$

where $F(\Gamma,\mu)$ is defined by Eq. (6.14) and $G(\Gamma,\mu)$ by Eq. (7.3). Consider first Eq. (A10). The two conditions defining the sector imply that the two first terms on the left-hand side are negligible compared to the last term. Therefore, if a solution exists in this sector we must have

$$F(\Gamma,\mu)=0. \tag{A12}$$

In Eq. (A11) it is clear that the q^2 term is negligible compared to the $q^{d+4-2\Gamma-\mu}$ term. Two possibilities seem to arise now. Either the last term on the right-hand side of the equation dominates the two other terms and then we must have $G(\Gamma,\mu)=0$ or the last and first terms are proportional to the same power of q, leading to the scaling relation $d+4-\Gamma-2\mu=0$. As discussed in the previous section the first possibility, which implies $d+4-\Gamma-2\mu<0$, is inconsistent with the whole idea of an expansion, where higher-order terms are not expected to be more violent than lower-order ones.

Points on the boundaries of the sectors are not possible solutions, basically because of the logarithmic factors that make power-law solutions or even solutions of the form $\phi_0 \propto q^{-\Gamma}[\ln(q_0/q)]^{\Gamma_1}$ and $\omega_0 \propto q^{\mu}[\ln(q_0/q)]^{\mu_1}$ inconsistent. The final conclusion is that only two possibilities exist. For $d > 2$ we find that the weak-coupling solution is possible. We find also a strong-coupling solution whenever the two equations $d+4-\Gamma-2\mu=0$ and $F(\Gamma,\mu)=0$ have a solution.

How does our discussion change if we use the fitting of the characteristic frequency (5.14) instead of the excitation energy (5.4)? Equation (6.1) is unchanged, but a term is added to Eq. (6.2). After separating high from low momenta we recover Eq. (6.11) and Eq. (6.12) changes into

$$(1+c)\omega_q-(\nu+A_3)q^2-J^{<}(q)-k_1^{<}(q)\omega_q+k_2^{<}(q)\omega_q\phi_q^{-1}=0, \tag{A13}$$

where $K_1^{<}(q)$ is given by

$$K_1^{<}(q)=\frac{2g^2}{(2\pi)^d}\frac{A}{B^2}q^{d+4-\Gamma-2\mu}\int^{q_0/q}d^dt \times\frac{\vec{t}\cdot(\hat{e}-\vec{t})[\hat{t}\cdot\hat{e}t^{-\Gamma}+(\hat{e}-\vec{t})\cdot\hat{e}|\hat{e}-\vec{t}|^{-\Gamma}]}{[t^{\mu}+|\hat{e}-\vec{t}|^{\mu}][t^{\mu}+|\hat{e}-\vec{t}|^{\mu}+1]} \tag{A14}$$

and

$$K_2^{<}(q)=\frac{2g^2}{(2\pi)^d}\frac{A^2}{B^2}q^{d+4-2\Gamma-2\mu}\int^{q_0/q}d^dt \frac{[\vec{t}\cdot(\hat{e}-\vec{t})]^2t^{-\Gamma}|\hat{e}-\vec{t}|^{-\Gamma}}{[t^{\mu}+|\hat{e}-\vec{t}|^{\mu}][t^{\mu}+|\hat{e}-\vec{t}|^{\mu}+1]}. \tag{A15}$$

A direct check of Eq. (A13) shows that the two last terms that were added to the left-hand side of the equation cannot dominate over the q^{μ} term. (In the case that $d+4-\Gamma-2\mu=0$, they have the same q dependence.) Therefore, we obtain exactly the same solution obtained for the case $K_1^{<}(q)=K_2^{<}(q)=0$ discussed before.

If we use a different expansion, by ascribing, say, M to *all* terms in $L-L_0$, the equations are changed by adding factors of q^2/ω_q to the nonintegral terms as obtained in Ref. [2]. The result is again the same. A weak-coupling solution may exist only for $d>2$ and the strong-coupling solution is determined by solving $d+4-\Gamma-2\mu=0$ and $F(\Gamma,\mu)=0$ simultaneously.

When considering integral equations of the form discussed above, it is usually expected that a power counting solution may be obtained by equating the power law arising from the integral part of the equation to some other term that is also a power law in q. Indeed, this is sometimes the case [20]. In our case the strong-coupling solution is of a different nature and the solution of a transcendental equation is required to fix the exponent.

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